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An extension of the Chowla–Selberg formula useful in quantizing with the Wheeler–De Witt equation

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Abstract. The two-dimensional inhomogeneous zeta-function series (with homogeneous part of the most general Epstein type) $\sum_{m,n \in \mathbb{Z}} (am^2 + bmn + cn^2 + q)^{-s}$ is analytically continued in the variable s using zeta-function techniques. A simple formula is obtained which extends the Chowla–Selberg formula to inhomogeneous Epstein zeta-functions. The new expression is then applied to solve the problem of computing the determinant of the basic differential operator that appears in an attempt at quantizing gravity by using the Wheeler–De Witt equation in $(2+1)$ -dimensional spacetime with the torus topology.

1. Introduction

In a recent publication [1], dealing with the approach to $(2+1)$ -dimensional quantum gravity which consists of making direct use of the Wheeler–De Witt equation, Carlip has come across a rather involved mathematical problem. Of course, none of the approaches that have been employed for the quantization of gravity is simple, for different reasons. Here we will concentrate only on the specific point of the whole problem that has been raised by Carlip, and which concerns the calculation of the basic determinant that appears in his method for the case of the torus topology. It is the determinant corresponding to a differential operator, D_0 , which has the following set of eigenfunctions and eigenvalues (for explicit details, see [1] and references therein):

$$|mn\rangle = e^{2\pi i(mx+ny)} \quad l_{mn} = \frac{4\pi^2}{\tau_2} |n - m\tau|^2 + V_0 \quad (1)$$

where m and n are integers, and τ and τ_2 are the usual labels corresponding to the standard two-dimensional metric for the torus

$$d\bar{s}^2 = \tau_2^{-1} |dx + \tau dy|^2 \quad (2)$$

with x and y angular coordinates of period 1 and $\tau = \tau_1 + i\tau_2$ the modulus (a complex parameter [1]). V_0 is the spatial integral of the relevant potential function [1].

At that point, the physical difficulty has boiled down to a well formulated mathematical problem which, unfortunately, has no straightforward solution from, for example, the zeta functions which commonly appear in physical or mathematical references, since, in fact, the best way to obtain the determinant once the spectrum of the operator is known is through

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the calculation of the corresponding zeta function, ζ_{D_0} . After simplifying the notation a little, one easily recognizes that one has to deal here with a series of the form

$$F(s; a, b, c; q) \equiv \sum'_{m,n \in \mathbb{Z}} (am^2 + bmn + cn^2 + q)^{-s} \tag{3}$$

the prime meaning that the term with both $m = n = 0$ is absent from the sum. Of course this distinction need not be made when $q \neq 0$ (the value of such term being then trivially q^{-s}), but it is certainly important for considering the particular case $q = 0$ (see later). One is interested in obtaining the function $F(1; a, b, c; q)$ of a, b, c, q , since this expression comes inside a functional integral which involves the relevant variables of the problem. As it stands, equation (3) has no sense for $s = 1$, and it is also clear that analytic continuation to such a value of s hits a pole and, therefore, must be conveniently defined. This has been done successfully in the literature (see [2]).

In what follows we will calculate the sum (3)—and its corresponding analytical continuation—and also its derivative with respect to s , by means of zeta-function techniques. The final expression will be remarkably simple, involving just (apart from finite sums) a quickly convergent series of exponentially vanishing integrals (of Bessel-function type). It is, in fact, a generalization of the celebrated (by mathematicians) Chowla–Selberg formula. We will proceed step by step, starting from some particular, more simple cases.

The case $b = 0$ will be treated in section 2, and the general homogeneous case ($q = 0$) in section 3—this is the Chowla–Selberg formula itself. In section 4 we will derive the new formula, which is capable of dealing with the general situation ($q \neq 0$). The explicit use of the formula in the quantization of $(2 + 1)$ -dimensional gravity will be treated in section 5, where specific results for this physical application will be given. For the sake of comparison, we present in section 6 an alternative treatment by means of Eisenstein series. Finally, section 7 is devoted to conclusions.

2. Case $b = 0$

Remember that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \zeta_H(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad a \neq 0, -1, -2, \dots \quad \text{Re } s > 1 \tag{4}$$

define the Riemann and Hurwitz zeta functions, respectively, which can be analytically continued in s as meromorphic functions to the whole complex s -plane (with just a simple pole at $s = 1$). In general, in zeta-function regularization we are finished when the result can be expressed in terms of these simple functions. However, in the more difficult case involved here, we will have a different function as the most elementary one (see (21) below).

Considering again the general series (3), the parenthesis in this expression must be visualized in an inhomogeneous quadratic form:

$$Q(x, y) + q \quad Q(x, y) \equiv ax^2 + bxy + cy^2 \tag{5}$$

restricted to the integers. We shall assume all the time that $b > 0$, that the discriminant

$$\Delta = 4ac - b^2 > 0 \tag{6}$$

and that q is such that $Q(m, n) + q \neq 0, \forall m, n \in \mathbb{Z}$. In terms of the corresponding physical constants, as they appear in [1], these conditions are indeed satisfied (for the physically relevant cases). We start by studying some particular situations which, for the benefit of the reader, may be interesting to recall.

The case $b = 0$ corresponds to a situation that we have considered in former papers (even in more general terms) and for which we have already derived explicit formulae [2] (see also [3]). In particular, for the series

$$E_2^c(s; a_1, a_2; c_1, c_2) \equiv \sum_{n_1, n_2=0}^{\infty} [a_1(n_1 + c_1)^2 + a_2(n_2 + c_2)^2 + c]^{-s} \tag{7}$$

we have obtained the following expression:

$$\begin{aligned} E_2^c(s; a_1, a_2; c_1, c_2) &= \frac{a_2^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(s+m)}{m!} \left(\frac{a_1}{a_2}\right)^m \zeta_H(-2m, c_1) E_1^{c/a_2}(s+m; 1; c_2) \\ &+ \frac{a_2^{1/2-s}}{2} \sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} E_1^{c/a_2}(s-\frac{1}{2}; 1; c_2) + \frac{2\pi^s}{\Gamma(s)} a_1^{-s/2-1/4} a_2^{-s/2+1/4} \\ &\times \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} n_1^{s-1/2} \cos(2\pi n_1 c_1) [a_2(n_2 + c_2)^2 + c]^{-s/2+1/4} \\ &\times K_{s-1/2} \left(\frac{2\pi n_1}{\sqrt{a_1}} \sqrt{a_2(n_2 + c_2)^2 + c} \right) \end{aligned} \tag{8}$$

where

$$\begin{aligned} E_1^c(s; a_1; c_1) &= \frac{c^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(s+m)}{m!} \left(\frac{a_1}{c}\right)^m \zeta_H(-2m, c_1) + \frac{c^{1/2-s}}{2} \sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \\ &+ \frac{2\pi^s}{\Gamma(s)} a_1^{-s/2-1/4} c^{-s/2+1/4} \sum_{n_1=1}^{\infty} n_1^{s-1/2} \cos(2\pi n_1 c_1) K_{s-1/2} \left(2\pi n_1 \sqrt{\frac{c}{a_1}} \right) \end{aligned} \tag{9}$$

with K_ν being the modified Bessel functions of the second kind. In order to specify the formula for the present case, we just have to substitute: $c_1 = c_2 = 1$ (the Hurwitz zeta functions simply turn into ordinary Riemann ones), $a_1 = a$, $a_2 = c$ and $c = q$, and be a bit careful with the summation range; but this can easily be taken care of (see, for instance, [4] where the explicit formulae relating the cases of doubly infinite summation ranges and simply infinite ranges are given).

3. Case $q = 0$

Here, the Chowla–Selberg formula [5] for the (general homogeneous) Epstein zeta function [6] corresponding to the quadratic form Q is to be used. (This is an expression well known in number theory [7] but not so much in mathematical physics.) The result is

$$\begin{aligned} F(s; a, b, c; 0) &= 2\zeta(2s) a^{-s} + \frac{2^{2s} \sqrt{\pi} a^{s-1}}{\Gamma(s) \Delta^{s-1/2}} \Gamma(s-\frac{1}{2}) \zeta(2s-1) + \frac{2^{s+3/2} \pi^s}{\Gamma(s) \Delta^{s/2-1/4} \sqrt{a}} \\ &\times \sum_{n=0}^{\infty} n^{s-1/2} \sigma_{1-2s}(n) \cos(n\pi b/a) \int_0^{\infty} dt t^{s-3/2} \exp \left[-\frac{\pi n \sqrt{\Delta}}{2a} (t + t^{-1}) \right] \end{aligned} \tag{10}$$

where

$$\sigma_s(n) \equiv \sum_{d|n} d^s \tag{11}$$

namely the sum over the s -powers of the divisors of n .

This formula is very useful and its practical application quite simple. In fact, the two first terms are just nice, while the last one (impressive in appearance) is very quickly convergent and thus absolutely harmless in practice: only a few first terms of the series need to be calculated, even if one needs exceptionally good accuracy. One should also notice that the pole of F at $s = 1$ appears through $\zeta(2s - 1)$ in the second term, while for $s = \frac{1}{2}$, the apparent singularities of the first and second terms cancel each other and no pole is formed.

A closer, quantitative idea about the integral can be got from the following closed expression for it:

$$\int_0^{\infty} dt t^{\nu-1} \exp\left(-\frac{\alpha}{t} - \beta t\right) = 2 \left(\frac{\alpha}{\beta}\right)^{\nu/2} K_{\nu}\left(2\sqrt{\alpha\beta}\right) \quad (12)$$

K_{ν} being again the modified Bessel function of the second kind. In particular, by calling the integral

$$I(n, s) \equiv \int_0^{\infty} dt t^{s-3/2} \exp\left[-\frac{\pi n\sqrt{\Delta}}{2a}(t + t^{-1})\right] \quad (13)$$

one has

$$\begin{aligned} I(n, 0) &= \sqrt{\frac{2a}{n\sqrt{\Delta}}} \exp\left(-\frac{\pi n\sqrt{\Delta}}{a}\right) = I(n, 1) & I(n, \tfrac{1}{2}) &= 2K_0(\pi n\sqrt{\Delta}/a) \\ I(n, 2) &= \frac{a + \pi n\sqrt{\Delta}}{\pi n\sqrt{\Delta}} \sqrt{\frac{2a}{n\sqrt{\Delta}}} \exp\left(-\frac{\pi n\sqrt{\Delta}}{a}\right) \\ I(n, 3) &= \frac{3a^2 + 3\pi n a\sqrt{\Delta} + \pi^2 n^2 \Delta}{\pi^2 n^2 \Delta} \sqrt{\frac{2a}{n\sqrt{\Delta}}} \exp\left(-\frac{\pi n\sqrt{\Delta}}{a}\right). \end{aligned} \quad (14)$$

As functions of n , all these expressions share the common feature of being exponentially decreasing with n .

4. The general case $a, b, c, q \neq 0$

This case is more difficult. To handle it, we can choose to go through the whole derivation of the Chowla–Selberg formula for the quadratic form Q and see the differences introduced by the inhomogeneity (the constant q). Instead, here we will undertake a more down-to-earth derivation, which will be similar to the ones that we have successfully employed several times in former papers—in particular to obtain (8) and (9). Since the technicalities of the method have been abundantly discussed before [2] (see also [4]), here we will consider the main steps of the proof only. They are the following. (i) Rewrite the initial expression (3), (8), by using the gamma-function identity

$$\sum_{m,n} (Q + q)^{-s} = \frac{1}{\Gamma(s)} \sum_{m,n} \int_0^{\infty} du u^{s-1} e^{-(Q+q)u}. \quad (15)$$

(ii) Expand the exponential in terms of power series of m and n and interchange the order of the summations, i.e. the sum over such expansion with the sums over m and n or—equivalently in this case—use Jacobi's theta-function fundamental identity (as can be found, for instance, in [8]). The equivalence of both methods was proven explicitly in [9]. The second one starts here from a trivial rewriting of the non-negative quadratic form

$Q(m, n)$ as the sum of two squares

$$Q(m, n) = a \left[\left(m + \frac{bn}{2a} \right)^2 + \frac{\Delta}{4a^2} n^2 \right] \tag{16}$$

and proceeds by considering the summation over m , while treating first n as a parameter.

(iii) Finally, make the following change of variables (for convenience):

$$u = \frac{2\pi m}{\sqrt{\Delta} n} t \tag{17}$$

and use the same idea as in (10) of rewriting the double sum as a sum over the product mn and (a finite one) over the divisors of the product

$$\sum_{n_1, n_2} \left(\frac{n_1}{n_2} \right)^{s-1/2} = \sum_{n_1, n_2} (n_1 n_2)^{s-1/2} n_2^{1-2s} = \sum_n n^{s-1/2} \sum_{d|n} d^{1-2s} \tag{18}$$

(this factor appears when the change of variables is performed). On the other hand, the term $\xi_n \equiv bn/(2a)$ in the first square of the decomposition (16) (that may be written $(m + \xi_n)^2 + n^2$) leads to a cosine factor in the final expression, e.g.

$$\cos(2\pi \xi_n) = \cos(n\pi b/a). \tag{19}$$

This is also explained in detail in [2] (but notice the small mistake in the first of these references, that was later corrected in the subsequent ones).

By doing all this, the following generalized expression is obtained

$$\begin{aligned} F(s; a, b, c; q) &= \sum_{m, n \in \mathbb{Z}} [Q(m, n) + q]^{-s} = \sum_{m, n \in \mathbb{Z}} (am^2 + bmn + cn^2 + q)^{-s} \\ &= 2\zeta_{\text{EH}}(s, 4aq/\Delta) a^{-s} + \frac{2^{2s} \sqrt{\pi} a^{s-1}}{\Gamma(s) \Delta^{s-1/2}} \Gamma(s - \frac{1}{2}) \zeta_{\text{EH}}(s - \frac{1}{2}, 4aq/\Delta) \\ &\quad + \frac{2^{s+3/2} \pi^s}{\Gamma(s) \Delta^{s/2-1/4} \sqrt{a}} \sum_{n=0}^{\infty} n^{s-1/2} \cos(n\pi b/a) \sum_{d|n} d^{1-2s} \\ &\quad \times \int_0^{\infty} dt t^{s-3/2} \exp \left\{ -\frac{\pi n \sqrt{\Delta}}{2a} \left[\left(1 + \frac{4aq}{\Delta d^2} \right) t + t^{-1} \right] \right\} \end{aligned} \tag{20}$$

where the function $\zeta_{\text{EH}}(s, p)$ (one-dimensional Epstein–Hurwitz or inhomogeneous Epstein) is given by

$$\begin{aligned} \zeta_{\text{EH}}(s; p) &= \sum_{n=1}^{\infty} (n^2 + p)^{-s} \\ &= -\frac{p^{-s}}{2} + \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{2 \Gamma(s)} p^{-s+1/2} + \frac{2\pi^s p^{-s/2+1/4}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n \sqrt{p}) \end{aligned} \tag{21}$$

and is studied in full detail in [10] (with numerical tables, plots, and a couple of explicit physical applications).

It is remarkable that the integral inside the series of the new expression can still be written in a closed form using (12)—as in the case of (10). Now using the integral

$$J(n, s) \equiv \int_0^{\infty} dt t^{s-3/2} \exp \left\{ -\frac{\pi n \sqrt{\Delta}}{2a} \left[\left(1 + \frac{4aq}{\Delta d^2} \right) t + t^{-1} \right] \right\} \tag{22}$$

we obtain, in particular,

$$\begin{aligned}
 J(n, 0) &= \sqrt{\frac{2a}{n\sqrt{\Delta}}} \exp \left[-\frac{\pi n}{a} \left(\Delta + \frac{4aq}{d^2} \right)^{1/2} \right] \\
 J(n, \frac{1}{2}) &= 2K_0 \left(\frac{\pi n}{a} \sqrt{\Delta + \frac{4aq}{d^2}} \right) \\
 J(n, 1) &= \sqrt{\frac{2a\sqrt{\Delta}}{n}} \left(\Delta + \frac{4aq}{d^2} \right)^{-1/2} \exp \left[-\frac{\pi n}{a} \left(\Delta + \frac{4aq}{d^2} \right)^{1/2} \right] \\
 J(n, 2) &= \left(\frac{a}{\pi n} + \sqrt{\Delta + \frac{4aq}{d^2}} \right) \sqrt{\frac{2a\Delta^{3/2}}{n}} \left(\Delta + \frac{4aq}{d^2} \right)^{-3/2} \exp \left[-\frac{\pi n}{a} \left(\Delta + \frac{4aq}{d^2} \right)^{1/2} \right] \\
 J(n, 3) &= \left(\frac{3a^2}{\pi^2 n^2} + \frac{3a}{\pi n} \sqrt{\Delta + \frac{4aq}{d^2}} + \Delta + \frac{4aq}{d^2} \right) \sqrt{\frac{2a\Delta^{5/2}}{n}} \left(\Delta + \frac{4aq}{d^2} \right)^{-5/2} \\
 &\quad \times \exp \left[-\frac{\pi n}{a} \left(\Delta + \frac{4aq}{d^2} \right)^{1/2} \right]
 \end{aligned} \tag{23}$$

which are again exponentially decreasing with n .

Expression (24) itself can also be written in terms of these Bessel functions:

$$\begin{aligned}
 F(s; a, b, c; q) &= 2\zeta_{\text{EH}}(s, 4aq/\Delta) a^{-s} + \frac{2^{2s} \sqrt{\pi} a^{s-1}}{\Gamma(s)\Delta^{s-1/2}} \Gamma(s - \frac{1}{2}) \zeta_{\text{EH}}(s - \frac{1}{2}, 4aq/\Delta) \\
 &\quad + \frac{8(2\pi)^s}{\Gamma(s)\Delta^{s-1/2}\sqrt{2a}} \sum_{n=0}^{\infty} n^{s-1/2} \cos(n\pi b/a) \sum_{d|n} d^{1-2s} \left(\Delta + \frac{4aq}{d^2} \right)^{s/2-1/4} \\
 &\quad \times K_{s-1/2} \left(\frac{\pi n}{a} \sqrt{\Delta + \frac{4aq}{d^2}} \right).
 \end{aligned} \tag{24}$$

Equation (24) is the fundamental result of this paper and must be given a name. We propose to call it *inhomogeneous* or *the generalized Chowla–Selberg* formula. To our knowledge, it has never appeared before in the mathematical (or physical) literature.

5. Explicit application of the formula in quantizing gravity through the Wheeler–De Witt equation

As discussed in [1], the quantization of gravity in 2+1 dimensions by means of the Wheeler–De Witt equation, in a spacetime with the topology $\mathcal{R} \times T^2$ (T^2 being the two-dimensional torus) of standard metric given by (2), proceeds through the calculation of the zeta function corresponding to the basic differential operator D_0 , which has a spectral decomposition given by (1). In terms of the function $F(s; a, b, c; q)$ (equation (3)), the zeta function of D_0 is

$$\zeta_{D_0}(s) = F \left(s; 4\pi^2/\tau_2, -8\pi^2\tau_1/\tau_2, 4\pi^2(\tau_1^2 + \tau_2^2)/\tau_2; V_0 \right). \tag{25}$$

One has, in particular, $\Delta = 64\pi^4$ and, using (24), one gets

$$\begin{aligned} \zeta_{D_0}(s) = & \frac{2^{-2s+1}\pi^{-2s}}{\tau_2^{-s}} \zeta_{\text{EH}}(s, V_0/(4\pi^2\tau_2)) \\ & + \frac{2^{-2s+1}\pi^{-2s+1/2}\Gamma(s-1/2)}{\tau_2^{s-1}\Gamma(s)} \zeta_{\text{EH}}\left(s-\frac{1}{2}, V_0/(4\pi^2\tau_2)\right) \\ & + \frac{2^{-2s+2}\pi^{-s}\sqrt{\tau_2}}{\Gamma(s)} \sum_{n=0}^{\infty} n^{s-1/2} \cos(2n\pi\tau_1) \sum_{d|n} d^{1-2s} \\ & \times \int_0^{\infty} dt t^{s-3/2} \exp\left\{-n\pi\tau_2 \left[\left(1 + \frac{V_0}{4\pi^2 d^2 \tau_2}\right) t + t^{-1} \right]\right\} \end{aligned} \tag{26}$$

with

$$\begin{aligned} \zeta_{\text{EH}}(s; V_0/(4\pi^2\tau_2)) = & -2^{2s-1}\pi^{2s} \left(\frac{V_0}{\tau_2}\right)^{-s} + 2^{2s-2}\pi^{2s-1/2} \frac{\Gamma(s-12)}{\Gamma(s)} \left(\frac{V_0}{\tau_2}\right)^{-s+1/2} \\ & + \frac{2^{s+1/2}\pi^{2s-1/2}}{\Gamma(s)} \left(\frac{V_0}{\tau_2}\right)^{-s/2+1/4} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(n\sqrt{V_0/\tau_2}). \end{aligned} \tag{27}$$

The quantity of interest is the determinant of the operator D_0 (see [1]). This is most conveniently computed by means of its zeta function. In particular:

$$\det^{1/2} D_0 = \exp\left[-\frac{1}{2}\zeta'_{D_0}(0)\right]. \tag{28}$$

Thus, we must now calculate the derivative of (24) at $s = 0$. We have, for the general function $F(s; a, b, c; q)$,

$$\begin{aligned} F'(0; a, b, c; q) = & \ln a + 2\zeta'_{\text{EH}}(0, 4aq/\Delta) - \frac{2\pi\sqrt{\Delta}}{a} \zeta_{\text{EH}}\left(-\frac{1}{2}, 4aq/\Delta\right) \\ & + 4 \sum_{n=1}^{\infty} n^{-1} \cos(n\pi b/a) \sum_{d|n} d \exp\left[-\frac{\pi n}{a} \left(\Delta + \frac{4aq}{d^2}\right)^{1/2}\right] \end{aligned} \tag{29}$$

where

$$\zeta'_{\text{EH}}(0; p) = -\pi\sqrt{p} + \frac{1}{2} \ln p + 2p^{1/4} \sum_{n=1}^{\infty} n^{-1/2} K_{1/2}(2n\pi\sqrt{p}) \tag{30}$$

while for $\zeta_{\text{EH}}(-\frac{1}{2}; p)$ the principal part prescription (PP) is to be used (see [2, 13, 14]):

$$\text{PP}\zeta_{\text{EH}}\left(-\frac{1}{2}; p\right) = -\frac{p}{4} - \frac{\sqrt{p}}{2} - \frac{\sqrt{p}}{\pi} \sum_{n=1}^{\infty} n^{-1} K_1(2n\pi\sqrt{p}). \tag{31}$$

Finally, for the determinant of D_0 , we obtain

$$\begin{aligned} \det^{1/2} D_0 = & \exp\left\{-\frac{1}{2} \ln a - \zeta'_{\text{EH}}(0; V_0/(4\pi^2\tau_2)) + 2\pi\tau_2 \zeta_{\text{EH}}\left(-\frac{1}{2}; V_0/(4\pi^2\tau_2)\right)\right. \\ & \left. - 2 \sum_{n=1}^{\infty} n^{-1} \cos(2n\pi\tau_1) \sum_{d|n} d \exp\left[-2n\pi\tau_2 \left(1 + \frac{V_0}{4\pi^2\tau_2 d^2}\right)^{1/2}\right]\right\} \end{aligned} \tag{32}$$

with $\zeta'_{\text{EH}}(0; V_0/(4\pi^2\tau_2))$ and $\zeta_{\text{EH}}(-\frac{1}{2}; V_0/(4\pi^2\tau_2))$ being given by expressions (30) and (31) above, putting $p = V_0/(4\pi^2\tau_2)$. This yields

$$\det^{1/2} D_0 = \frac{\tau_2}{\sqrt{V_0}} \exp \left\{ -\frac{V_0}{8\pi} + \frac{1}{2} \sqrt{\frac{V_0}{\tau_2}} - \frac{1}{2} \sqrt{\tau_2 V_0} \right. \\ \left. - \sqrt{\frac{2}{\pi}} \left(\frac{V_0}{\tau_2} \right)^{1/4} \sum_{n=1}^{\infty} n^{-1/2} K_{1/2} \left(n \sqrt{\frac{V_0}{\tau_2}} \right) - \frac{1}{\pi} \sqrt{\tau_2 V_0} \sum_{n=1}^{\infty} n^{-1} K_1 \left(n \sqrt{\frac{V_0}{\tau_2}} \right) \right. \\ \left. - 2 \sum_{n=1}^{\infty} n^{-1} \cos(2n\pi\tau_1) \sum_{d|n} d \exp \left[-2n\pi\tau_2 \left(1 + \frac{V_0}{4\pi^2\tau_2 d^2} \right)^{1/2} \right] \right\}. \quad (33)$$

We observe again that the final formula is really simple since, in practice, it provides a very good approximation with just a few terms, which are, in turn, elementary functions of the relevant variables and parameters. This is so, because the infinite series that appear converge extremely quickly (terms exponentially decreasing with n). In an asymptotic approach to the determinant, only the first line in (33) is relevant (as we will show) and the three series can be eliminated altogether.

From the detailed analysis in [1], it follows that the quantity to be calculated now is the derivative with respect to V_0 of the above determinant, since this quantity vanishes precisely at the solutions of the Hamiltonian constraint (always in the language of quantization through the corresponding Wheeler–De Witt equation). In other words, the solutions of the equation

$$\frac{\partial}{\partial V_0} \det^{1/2} D_0 = 0 \quad (34)$$

will yield the conditions that the quantized magnitudes and parameters are bound to satisfy as a consequence of the Wheeler–De Witt equations. If this does not provide all the solutions of such (very involved) differential equations, at least it gives us important clues about their behaviour (this is one of the two basic problems of the approach in [1], namely that of understanding the determinant of D_0). After some calculations one finds that (34) can be written as

$$\det^{1/2} D_0 \left[-\frac{1}{8\pi} - \frac{1}{2V_0} - \frac{1}{4} \sqrt{\frac{\tau_2}{V_0}} + \frac{1}{4\sqrt{\tau_2 V_0}} - \frac{(\tau_2/V_0)^{3/4}}{2\sqrt{2\pi}\tau_2} \sum_{n=1}^{\infty} n^{-1/2} K_{1/2} \left(n \sqrt{\frac{V_0}{\tau_2}} \right) \right. \\ \left. - \frac{1}{2\pi} \sqrt{\frac{\tau_2}{V_0}} \sum_{n=1}^{\infty} n^{-1} K_1 \left(n \sqrt{\frac{V_0}{\tau_2}} \right) - \frac{(\tau_2/V_0)^{1/4}}{\sqrt{2\pi}\tau_2} \sum_{n=1}^{\infty} n^{1/2} K_{1/2}' \left(n \sqrt{\frac{V_0}{\tau_2}} \right) \right. \\ \left. - \frac{1}{2\pi} \sum_{n=1}^{\infty} K_1' \left(n \sqrt{\frac{V_0}{\tau_2}} \right) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \cos(2n\pi\tau_1) \sum_{d|n} \left(d^2 + \frac{V_0}{4\pi^2\tau_2} \right)^{-1/2} \right. \\ \left. \times \exp \left(-2n\pi\tau_2 \sqrt{1 + \frac{V_0}{4\pi^2\tau_2 d^2}} \right) \right] = 0 \quad (35)$$

(here the primes mean derivatives of the Bessel functions). In principle—the consistency of the approximation is to be checked *a posteriori*—equation (35) can be reduced to the very simple expression

$$\frac{1}{2\pi} \frac{V_0}{\tau_2} + \left(1 - \frac{1}{\tau_2} \right) \sqrt{\frac{V_0}{\tau_2}} + \frac{2}{\tau_2} \simeq 0. \quad (36)$$

The analysis of this last equation is easy to do. From its discriminant it turns out that real solutions can only be obtained when

$$\tau_2 \leq \tau_2^{(1)} \quad \text{or} \quad \tau_2 \geq \tau_2^{(2)} \tag{37}$$

with

$$\begin{aligned} \tau_2^{(1)} &\equiv 1 + \frac{2}{\pi} \left(1 - \sqrt{\pi + 1}\right) = 0.341\,041\,03 \\ \tau_2^{(2)} &\equiv 1 + \frac{2}{\pi} \left(1 + \sqrt{\pi + 1}\right) = 2.932\,198\,52. \end{aligned} \tag{38}$$

Moreover, the special situation when $\tau_2 = 1$ leads to the non-physical result $V_0 = -4\pi$. Now, from the constraint $\sqrt{V_0/\tau_2} > 0$ (in order for the whole approximation to have sense) we see that the second possibility in (37) just disappears. For the first, when $\tau_2 \rightarrow 0$ it turns out that $\sqrt{V_0/\tau_2} \sim 2\pi/\tau_2 \rightarrow \infty$ or $\sqrt{V_0/\tau_2} \rightarrow 2$. When τ_2 moves within the allowed range $0 < \tau_2 \leq \tau_2^{(1)}$ (the first interval of (37)), the corresponding two solutions $\sqrt{V_0/\tau_2}$ of the quadratic equation (36) sweep the following intervals

$$0 < \tau_2 \leq \tau_2^{(1)} = 0.34 \longrightarrow \begin{cases} +\infty > \sqrt{V_0/\tau_2} \geq 2\pi \frac{\sqrt{\pi+1}-1}{\pi+2-2\sqrt{\pi+1}} = 6.07 \\ 2 < \sqrt{V_0/\tau_2} \leq 2\pi \frac{\sqrt{\pi+1}-1}{\pi+2-2\sqrt{\pi+1}} = 6.07 \end{cases} \tag{39}$$

in the order indicated, respectively. It is easy to check that the first of these two alternatives (corresponding to the bigger root of (36)) provides an absolutely consistent approximate solution to the exact equation (35). On the contrary, the second alternative (corresponding to the smaller root of (36)) does not actually provide a consistent approximate solution (unless τ_2 is close to $\tau_2^{(1)}$). Direct investigation of other possible roots of (35) (e.g. involving some terms of the series) is a quite difficult issue.

6. An alternative treatment by means of Eisenstein series

An alternative way of treating the general case is the following (see [1]). The inhomogeneity (the q -term here) is taken care of by the simplest (but hardly economic) method of performing a binomial expansion of the sort [14]

$$\sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{k! \Gamma(s)} q^k E(z, s+k) \tag{40}$$

where $E(z, s)$ is an Eisenstein series (see, for instance, Lang [11] or Kubota [12]), which is obtained from $F(s; a, b, c; 0)$ by doing the substitution

$$2z = a + iu \quad c = C \frac{u}{2} \tag{41}$$

so that

$$E(z, s) = \sum_{m,n=0}^{\infty} (u/2)^s |m + nz|^{-2s} \tag{42}$$

and has the series expansion

$$\begin{aligned} E(z, s) &= 2\zeta(2s) + 2\sqrt{\pi} (u/2)^{1-s} \frac{\Gamma(1-s/2)\zeta(2s-1)}{\Gamma(s)} \\ &+ 2 \sum_{m=1}^{\infty} \sum_{n \neq 0} e^{imna} \left(\frac{2|n|}{mu}\right)^{s-1/2} K_{s-1/2}(\pi mnu/2). \end{aligned} \tag{43}$$

It is important to notice, however, that when doing things in this last way the final result is expressed in terms of *three* infinite sums, while in the first general procedure only *one* infinite sum appears (together with a finite sum, for every index n , over the divisors of n), and it is very quickly convergent. Notice, moreover, how the d -term in the exponent in (24), when expanded in power series, gives rise to the binomial sum corresponding to the last treatment. The advantage of using the method developed in section 5, stemming from (24), seems clear (expanding a negative exponential is, in general, computationally disastrous).

7. Conclusions

The main results of this paper have been the derivation of (24) which generalizes the Chowla–Selberg formula (10) and its physical application to calculate the determinant (32) and its derivative (equations (35) and (36)). To have at our disposal an exact expression for dealing with inhomogeneous Epstein zeta functions is certainly an interesting thing from the physical point of view, since these kind of zeta functions appear frequently in modern applications that are no longer restricted to zero temperature or Euclidean spacetime. The simplicity of the result is remarkable. Namely, the fact that, in practice, we just need a couple of terms of (24) even if we want to obtain very accurate numerical results.

Two problems were singularized out in [1] as the main difficulties that appear in the quantization of (2+1)-dimensional gravity through the Wheeler–De Witt equation: (i) to give grounds for the choice of the specific operator ordering of the Hamiltonian constraint which leads to the Wheeler–De Witt equation of the quantized system, and (ii) to understand the functional dependence of the determinant $\det^{1/2} D_0$ in terms of the relevant variables and to obtain its extrema as a function of the potential V_0 . In the present paper we have been able to solve problem (ii)—at least partially—by means of a numerically consistent approximation. Furthermore, we now also have the possibility, through (35) and (36), of getting relevant hints which could lead to the solution of problem (i)—at least in a crude way. In the sense that if, once analysed in detail, the constraints (39) turned out to be unphysical, that should lead us to conclude that probably the operator ordering chosen in the Hamiltonian constraint (and thus the Wheeler–De Witt equation itself which emerges from it) ought to be modified.

Also to be noticed is the more technical point that, in general, when performing the analytic continuation through s (necessary, e.g., for the calculation of determinants of differential operators), or when taking the derivative of the series function F with respect to s at some particular value of s , it may well happen that we hit a pole since, in general, this continuation is a meromorphic function of s . That occurs in our case for $s = 1$. Such a situation can be dealt with in the usual way, by means of the principal-part prescription [13, 14]. In [14] (see also [2]), we discuss several explicit examples (appearing in physical theories) where the precise manner of doing so is clearly explained, all the way down to the numerical results (see also [10]).

Apart from the physical application that we have considered here—dealing directly with the generalization of the Chowla–Selberg formula derived above—in general our new expression will certainly be useful in physical situations involving massive theories, finite temperatures or a chemical potential in a compactified spacetime. This is the physical meaning to be attributed to the constant q . On the contrary, for mathematical uses in number theory the consideration of inhomogeneous quadratic forms for defining zeta functions does not seem to be especially relevant. This would explain why such formulae are not to be found in the mathematical literature, its derivation being, however, anything

but straightforward.

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